# Conditional averaging of a turbulent free surface

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A fully nonlinear dynamical boundary condition on a free surface in viscous turbulent flow is considered. Conditional averaging with a fixed normal to a free surface in a point revealed simple statistical structures for the first- and second-order moments of the deformation rate tensor. These structures persist in a free-surface boundary layer. The exact results presented suggest a different direction for experimental and numerical studies of free-surface turbulence.

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Free-surface turbulent flow is one of the most complex natural phenomenon with many applications. The important part of the free-surface turbulence is a statistical influence of the fully nonlinear dynamical boundary condition. For a clean surface (without surfactants, producing additional stress) this boundary condition on free surface has the form [1]

$$2\nu D_{ij} n_j = P_* n_i \ , \tag{1}$$

$$D_{ij} = \frac{1}{2}\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right] , \quad D_{ii} = 0 \ , \tag{2}$$

$$P_* = P + \tau\left[\frac{1}{R_1} + \frac{1}{R_2}\right] , \quad \frac{1}{R_1} + \frac{1}{R_2} = -\mathrm{div}\mathbf{n} \ . \tag{3}$$

Here $\nu$ is kinematic viscosity, $D_{ij}$ is deformation rate tensor for incompressible flow, $V_i$ is velocity field, $P$ is kinematic pressure (including gravitational effect), $\tau$ is surface-tension coefficient, $R_1$ and $R_2$ are principal radii of curvature at a given point of the surface, and $n_j$ is a unit normal to the free surface. According to (1), $n_i$ is the eigenvector of $D_{ij}$ and the corresponding eigenvalue is proportional to the modified pressure $P_*$.

One general method of analyzing nonlinear phenomena is conditional averaging. This method was applied to the Navier-Stokes equation, written in terms of a vorticity field [2–5]. Equation (1) begs for conditional averaging with fixed $n_i$ at a point:

$$\overline{D}_{ij} n_j = a n_i \ , \quad 2\nu a \equiv \overline{P}_* \ , \quad \overline{D}_{ii} = 0 \ . \tag{4}$$

Here the overbar means conditional averaging and it is not extended over $n_i$ because $n_i$ is fixed.

Now consider turbulent flow with a free surface, homogeneous and isotropic in the horizontal directions. A conditionally averaged scalar field, taken at the free surface, will depend only on time (for nonstationary turbulence) and one scalar argument $\mu \equiv n_i z_i$, where $z_i$ is a unit vertical vector. The parameter $\mu$ is a measure of the local steepness of the free surface.

For conditionally averaged tensor fields we have two distinguished vectors $n_i$ and $z_i$. Thus, taking into account symmetry (2), we have the general expression

$$\overline{D}_{ij} = a_1 n_i n_j + a_2 z_i z_j + a_3 (n_i z_j + n_j z_i) + a_4 \delta_{ij} \ , \tag{5}$$

where scalars $a_m$ depend only on time and $\mu$, and $\delta_{ij}$ is the unit tensor. By using Eqs. (4), we express $a_1$, $a_3$, and $a_4$ in terms of $a_2$ and, after simple algebra, get

$$\overline{D}_{ij} = a(2n_i n_j + y_i y_j - \delta_{ij}) + b(2y_i y_j + n_i n_j - \delta_{ij}) \ , \tag{6}$$

$$y_i = \frac{z_i - \mu n_i}{(1-\mu^2)^{1/2}} \ , \quad b = \tfrac{1}{2}(1-\mu^2) a_2 - \tfrac{1}{2} a \ . \tag{7}$$

Here, instead of $z_i$, we introduce the unit vector $y_i$, which is normal to $n_i$ and belongs to the vertical plane, defined by vectors $(\mathbf{z}, \mathbf{n})$. From (6) and (7) we see that $y_i$ is a different eigenvector of $\overline{D}_{ij}$ with $b$ as an eigenvalue:

$$\overline{D}_{ij} y_j = b y_i \ . \tag{8}$$

Now we can introduce the third unit eigenvector normal to $n_i$ and $y_i$:

$$s_i = \epsilon_{ijk} n_j y_k \ , \tag{9}$$

where $\epsilon_{ijk}$ is a unit antisymmetric tensor. Incompressibility (4) requires that the third eigenvalue is $-(a+b)$. Taking into account the identity

$$n_i n_j + y_i y_j + s_i s_j = \delta_{ij} \ , \tag{10}$$

Eq. (6) can be written in a canonical form

$$\overline{D}_{ij} = a n_i n_j + b y_i y_j - (a+b) s_i s_j \ . \tag{11}$$

Equations (6) and (11) represent a statistical structure on the turbulent free surface, revealed by conditional averaging. The deformation rate tensor generally has five independent components, which are random functions of coordinates and time and so are eigenvectors. After conditional averaging, the eigenvectors are fixed and the description of $\overline{D}_{ij}$ reduces to two scalars, which for statistically stationary turbulence depend only on one scalar argument $\mu$. The next step is to determine these two functions, which may parametrically depend also on Reynolds and Froude numbers. This principally can be done by optical measurements (with tracers and colored reflections [6]) and by direct numerical simulations (see, for example, the recent work [7]). On the other hand, the structure (6) and (11), being an exact result, can serve as a control of measurements and numerical experiments.

We can expect that this structure will only slowly change when we move from the free surface down a distance of the order of the Kolmogorov internal scale $l_\nu = \nu^{3/4}\epsilon^{-1/4}$ ($\epsilon$ is the mean dissipation rate of energy). In fact, at a point at a depth $d$ from the level of the equilibrium free surface we have the same general formula (5) for the conditionally averaged deformation rate tensor with fixed $\mathbf{n}$ in the corresponding point on the free surface (on the same vertical). Scalars $a_m$ will depend additionally on $d/l_\nu$. Let us stress that in conditional averaging we fix only the direction $\mathbf{n}$. It is conceivable in the future to fix also the local curvature of the free surface, which would yield another relevant length scale in addition to $l_\nu$. The incompressibility condition (4) reduces the number of independent scalars to 3. With the increase of $d$, the eigenvector $s_i$ apparently remains the same, but two other eigenvectors may rotate in the plane $(\mathbf{z},\mathbf{n})$.

Asymptotically, when the dependence of $\mathbf{n}$ is weak, the tensor becomes axisymmetric:

$$\bar{D}_{ij} = \tfrac{1}{2}\lambda(3z_iz_j - \delta_{ij}) . \quad (12)$$

Here $\lambda$ is an eigenvalue in the vertical direction; the eigenvalue in any horizontal direction is $-\lambda/2$. Let us note that an unconditionally averaged traceless tensor has the form (12) on any level $d$ because turbulence is axisymmetric. However, for the first-order moment of $D_{ij}$, the unconditional averaging gives zero. Indeed, by averaging (12) over $\mathbf{n}$ we have

$$\langle D_{ij} \rangle = \tfrac{1}{2}\langle \lambda \rangle (3z_iz_j - \delta_{ij}) ,$$

$$-\langle \lambda \rangle = 2\langle D_{11} \rangle = 2\left\langle \frac{\partial V_1}{\partial x_1} \right\rangle = 2\frac{\partial}{\partial x_1}\langle V_1 \rangle = 0 .$$

Here $\langle\ \rangle$ means unconditional averaging, $x_1$ is one of the horizontal coordinates, and we used horizontal homogeneity. Having in mind that the conditional averaging with a weak $\mathbf{n}$ dependence is close to unconditional averaging, we can expect that the axisymmetric regime is more pronounced for moments of $D_{ij}$ with nonzero unconditional averages.

Let us consider a second-order moment, which is relevant to the dissipation of the Reynolds stress tensor. Multiplication of (1) by $D_{ki}$ and conditional averaging gives

$$\overline{D_{ki}D_{ij}}n_j = An_k , \quad (2\nu)^2 A = \overline{P_*^2} . \quad (13)$$

Using a representation of the form (5) and (13), we get a formula similar to (11):

$$\overline{D_{ki}D_{ij}} = An_kn_j + By_ky_j + Cs_ks_j . \quad (14)$$

The difference is that the trace of (14) is positive, representing the conditionally averaged rate of the energy dissipation. The structure (14) with the rotation of two eigenvectors (see above) will again persist in the turbulent free-surface boundary layer.

We hope that the approach to the problem of free-surface turbulence presented herein will stimulate corresponding experimental and numerical studies.

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